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LOCAL REACHABILITY FOR DIFFERENTIAL CONTROL SYSTEMS WITH BANACH-VALUED TRAJECTORIES

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Local reachability for differential control systems with Banach-valued trajectories.

Introduction. In systems of differential equations depending on "controls", (parameters being functions of time), it is of importance to know if nearby points of the end point of a trajectory (solution) can be reached, by small variations of the controls.

This problem arises for example when one wants to establish Pontryagin maximum principles and [1], [2], [3], [4] contain such results implicitely in their proofs. Explicitely such results have been stated for example in [5], Ch.6, [6] for $\mathbb{R}_n$-valued trajectories. Below we give results on local reachability for switching-closed systems in the case of Banach valued trajectories. At the end of the paper we indicate how this result implies a maximum principle for such systems.

Definitions. X is a Banach space, let $J = [0,1] \subset \mathbb{R}$ have the Lebesque measure. For $p \in [1,\infty]$, $\mathcal{L}_p(J,X)$ is the set of Lebesque-measurable functions $f(.)$ such that $\|f(.)\|^p$ is integrable, or, for $p = \infty$, $\|f(.)\|$ is essentially bounded. (Measurability in the Bochner sence, [7].) $L_p(J,X)$ for $p \in [1,\infty]$ are the corresponding quotient spaces. Their elements are written $f(\ )$. The norms in these spaces are written $\|f\|_p$. $\mathcal{D}(J,X)$ is the set of maps $x(.) : x(t) = x + \int_0^t g$, for $g \in \mathcal{L}_1(J,X)$, $x \in X$. As a subset of $\mathcal{C}(J,X)$ (the continuous maps $J \to X$), it is normed by the supremumsnorm $\| \|^{\infty}$. Continuity and continuous differentiability with respect to two metrics α and β (in the domain and range spaces, resp.) is written continuity (α,β), continuous differentiability (α,β). Other topological concepts involving one or both norms are written similarly (e.g. convergence (α),): with respect to α). In a product of two spaces the product metric is denoted by $\alpha \times \beta$. An open neighborhood of a subset (or point) x with "radius" δ is written $B(x,\delta)$. An error function $e(d)$ is an extended realvalued nonnegative function on $\langle 0,\infty \rangle$ such that $\lim_{d \to 0+} e(d)$ exists and equals zero. If Z,Z' are normed spaces, $\mathcal{L}(Z,Z')$ denotes the set of continuous linear maps from Z into Z'. A map $g(.) : Z \to Z'$ is continuously differentiable on a convex subset $A \subset Z$ if there is a continuous map $g'(.) : A \to \mathcal{L}(Z,Z')$ 1), and, for each a_0 in A an errorfunction $e(d)$ such that

$$\|g(a)-g(a_0)-g'(a_0)[a-a_0]\| \leq e(d)\cdot\|a-a_0\|, \quad a \in B(a_0,d)$$

Usual properties hold also for this definition of continuous differentiation, in particular

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1) this set topologized by the supremumsnorm.

$$(1)\quad \| g(a')-g(a)-g'(a_0)[a'-a]\| \leq \sup_{s\in[a':a]} \|g'(s)-g'(a_0)\| \cdot \|a'-a\|$$

for $a',a,a_0 \in A$. See [8] Ch.VIII.

Let I be some set. A subset $\mathcal{F}$ of $X^{I\times J}$ is said to have property (SW) iff:
$g\in\mathcal{F}$, $g'\in\mathcal{F}$; $M\subset J$, M measurable $\Rightarrow$ $g\cdot M+g'(J-M)\in\mathcal{F}$
(we apply the symbol of a set also as a symbol of its indicator function). $\mathcal{F}$ has property ($c\sigma$) iff it is closed in $X^{I\times J}$ in the invariant pseudometric given by:

$$\sigma(f,0) = \inf\{\mathrm{meas}(M)/M \supset \{t/f(i,t)\neq 0\}, \forall i\in I\}$$

(the M's being measurable).

<u>Linear differential equations</u>. Let $A(t)\in\mathcal{L}(X)=\mathcal{L}(X,X)$, for $t\in J$, $A(.)\in\mathcal{L}_\infty(J,\mathcal{L}(X))$. Then the equation

$$(2)\qquad \dot{x}(t) = A(t)[x(t)]+g \quad \text{a.e. in } J,\ x(t) = x_0+\int_0^t \dot{x}(.)$$

has a a.e. unique solution $\dot{x}_g(.)$ for each $g\in\mathcal{L}_1(J,X)$, x_0 arbitrary in X. $g\to\dot{x}_g(\ )$ is continuous $(\|\ \|_1, \|\ \|_1)$.

The equation $U'(t) = A(t)\circ U(t)$ has a a.e. unique solution $\dot{C}_v(t)\in\mathcal{L}_\infty(J,\mathcal{L}(X))$ on J, such that $C_v(v) = I$, $v\in J$ and, if $C_1(t)^{-1}= C(t)$;[1] $x_g(1) = \int_0^1 C(s)\cdot g(s)ds$.
$C_v(s)$ is called the "resolvent" of eq. (2). See [8]

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[1] $\forall t\in J$, $C_v(t)^{-1}$ exists as an element of $\mathcal{L}(X)$

Chp. X, (and [9] Ch. 3 Probl.1., also valid in Banach space).

<u>Local reachability</u>. Definitions. Let $\mathcal{F}$ be a subset of $X^{X\times J}$. For $x(.)\in\mathcal{D}(J,X)$ define the property

(Rg) : There are constants M and δ, both >0 such that for all $f"\in\mathcal{F}$ and all $x\in B(x(J),\delta)$, the following holds : $\| f"(x,t)\| \leq M, \forall t\in J$, $f"_x(x,t)$ exists continuously at x, and $\| f"_x(x,t)\| \leq M$, both for all $t\in J$; and, finally, $f(x,.)\in\mathcal{L}_\infty(J,X)$, $f_x(x,.)\in\mathcal{L}_\infty(J,\mathcal{L}(X))$.

Let $x_o\in X$, Call pairs $(\dot{x}(.),f)\in\mathcal{L}_1(J,X)\times\mathcal{F}$ fulfilling

$$(3)\qquad \dot{x}(.) = f(x(.).)\ \text{a.e. on}\ J, x(t) = x_o+\int_o^t \dot{x}(.)$$

system pairs[1]. A point $x\in X$ is said to bereachable if there is a system pair $(\dot{x}(.),f)$ for which $x(1) = x$.

Let $(\dot{\bar{x}}(.),\bar{f})$ be a fixed system pair such that (Rg) holds (for $\bar{x}(.)$). Let $A(.)$ of eq. (2) be the map $\bar{f}_x(\bar{x}(.),.)$, and denote the solutions $\dot{x}_g(.)$ of (2), for $g = f(\bar{x}(.),.)-\bar{f}(\bar{x}(.),.)$, by $\dot{q}_f(.)$. Then

$$(4)\qquad q_f(1) = \int_J C(s)\cdot(f(\bar{x}(.),.)-\bar{f}(\bar{x}(.),.))d\mu$$

1) thus, implicitely assumed, such pairs $(\dot{x}(.),f)$ have the property that $f(x(.),.)$ is measurable.

Finally, let $cB(x,\delta) = \text{int co}\{0, B(x,\delta)\}$, for balls $B(x,\delta)$ in X. Now we can state our main result.

Theorem: Let $\mathcal{F} \subset X^{X\times J}$ have the properties (SW) and ($c\sigma$) (for $I = X$), let $(\bar{x}(.),\bar{f})$ be a system pair such that (Rg) holds for $\bar{x}(.)$. Then, if $cB(\bar{p},2\bar{e}) \subset \overline{co}\{q_f(1)/f \in \mathcal{F}\} = K$ where $\bar{e} > 0$, $\bar{p} \in X$, there is a $d > 0$ such that all points of $cB(d\bar{p},d\bar{e})+\bar{x}(1)$ are reachable.

We start the proof by considering solutions (pairs) for f near $\bar{f}$. Let $B = B(\bar{x}(J),\delta)$, $B_1 = B(\dot{\bar{x}}(\ ),\delta)$, $B' = B(\bar{x}(.),\delta) \subset \mathcal{D}(J,X)$, and norm $\bar{\mathcal{F}} = \text{linspan}\,\mathcal{F}$ by $\|f\|^+ = \max\{\sup_{z(\cdot)\in B'} \|f(z(\cdot),.)\|_1, \sup_{z(\cdot)\in B'} \|f_x(z(.),.)\|_1$. By aid of the continuous differentiability in B, we may easily establish the following properties of the map $F : B_1 \times \text{co}\,\mathcal{F} \to L_1(J,X)$, $F(\dot{x}(\ ),f) = f(x(\ ),\ ),(x(t) = x_o + \int_0^t \dot{x}(.))$: F has a partial derivative F_f at $\dot{x}(\ )$ equal to the linear map $\widetilde{x(.)} : f \to f(x(\ ),\ )$; $F_f(.,.)$ exists and is continuous $(\|\ \|_1 \times \|\ \|^+, \|\ \|_1)$ in $B_1 \times \text{co}\,\mathcal{F}$. The map $A : s(\ ) \to f_x(x(\ ),\ )[\int_0^{\cdot} s(\ )]$ is a partial derivative $F_{\dot{x}(\ )}$ of F at $\dot{x}(\ ) \in B_1$; being continuously $(\|\ \|_1 \times \|\ \|^+, \|\ \|_1)$ dependent on $(\dot{x}(\ ),f) \in B_1 \times \text{co}\,\mathcal{F}$.

By eq. (2), we get that $(I-A)^{-1}$ exists as an element of $\mathcal{L}(L_1(J,X))$. Then, by the implicit function theorem (a slight extension of [8] 10.2.1), there is a convex neighborhood N of $\bar{f}$ in co $\mathcal{F}$ and a map $f \to \dot{x}_f(\ )$ such that $\dot{x}_{\bar{f}}(\ ) = \dot{\bar{x}}(\ )$ and $(\dot{x}_f(\ ),f)$ fits in (3) for all $f \in N$. $\dot{x}_f(\ )$ is furthermore continuously differentiable $(\|\ \|^+, \|\ \|_1)$ in N, and at $\bar{f}$ the derivative is $(I-\bar{A})^{-1} \circ \widetilde{x(.)}$, $\bar{A} = A$ evaluated at

$(\dot{\bar{x}}(.),\bar{f})$. Thus also $y(f) = x_f(1)$ is continuously differentiable in N, and its derivative $y'(g)$ at $\bar{f}$ is given by

$$(5) \qquad y'(g) = \int_J C(s)g(\bar{x}(.),.)ds = q_{g(\bar{x}(.),.)}(1), \; g \in \bar{\mathcal{F}}$$

Thus from (1) we get, for $e(d) = \sup_{f \in B^+(\bar{f},d) \cap N} \| y'(f)-y'(\bar{f})\|$

$$(6) \|y(f')-y(f)-y'(\bar{f})[f'-f]\| \leq e(d)\cdot \|f'-f\|^+, \text{for } f',f \in B^+(\bar{f},d)\cap N.$$

By continuity of y', $e(.)$ becomes an error function.
Now we need a technical result, being a sort of generalization of the nonlinear interior mapping theorem [10].

Proposition. (Local reachability.)

Let Y be a normed space, let A be a complete pseudometric space. Let $\check{e}(d)$ be an errorfunction. Let $\bar{e} > 0$, $(\bar{e} \in \mathbb{R})$, $\bar{a} \in A$ and $\bar{p} \in Y$. For each $d \in \langle 0,d_o]$, $d_o > 0$, let A_d be a subset of A, $\bar{a} \in A_d$ for all d, and $A_d \subset A_{d'}$ if $d \leq d'$. Let $y(.) : \bar{A}_{d_o} \to Y$, $y'(.) : A \to Y$, define $\bar{y}(a) = y(a)-y(\bar{a})$. Let $y'(\bar{a}) = 0$ and let $y(.)$ be continuous. Assume for all $d \in \langle 0,d_o]$ that:

(A) $\quad \operatorname{diam}(A_d) \leq M.d$, M a constant > 0.

(B) $\quad$ For all $a,a'' \in A_d$, $k \in [0,1]$, there exists, for each $\varepsilon > 0$, an $a' \in A_d$ such that

(B_1) $\|ky'(a'')+(1-k)y'(a)-y'(a')\| \leq \varepsilon$, and

(B_2) $d(a,a') \leq Mkd$.

(C) $dy'(A) \subset \overline{y'(A_d)}$

(D) $\|y(a')-y'(a')-(y(a)-y'(a))\| \leq \check{e}(d)\cdot d(a',a)$, $a',a \in A_d$

Then if $\overline{y'(A)} \supset cB(2\overline{e},\overline{p})$, there is a $d' \in \langle 0,d_o]$ such that $cB(\overline{e}d,d\overline{p})+y(\overline{a}) \subset y(\overline{A}_d) \subset y(A)$ for all $d \in \langle 0,d']$.

Proof. Choose d' so that for all $d \in \langle 0,d']$ $\check{e}(d)\cdot M < {}^{e}/4$, $e = \overline{e}/3$. By (C) for each $p \in B(\overline{e},\overline{p})$ and $d \in \langle 0,1]$:

($\acute{C}$): $cB(d\overline{e},dp) \subset cB(d2\overline{e},d\overline{p}) \subset \overline{d1y(A)} \subset \overline{1y(A_d)}$

If we prove that $dp \in \overline{y}(A_d)$ for each $d \in \langle 0,d']$ and each $p \in B(\overline{e},\overline{p})$, then kdp is element of $\overline{y}(\overline{A}_{kd}) \subset \overline{y}(\overline{A}_d)$ for $k \in \langle 0,1]$, and hence $cB(d\overline{e},d\overline{p}) \subset \overline{y}(\overline{A}_d)$.

We shall now prove that $dp \in y(\overline{A}_d)$ for all $d \in \langle 0,d']$, $p \in B(\overline{e},\overline{p})$, and to this end we shall apply an induction process of successive "convex" approximations. In the induction step we shall use the following

<u>Sublemma</u>. Define $\check{y}(a)=dp+y'(a)-\overline{y}(a)$. Let $d(z)$ for $z \in Y$ mean the distance from z to $\complement B(ed,dp)$. We then have:

For each $a \in A_d$, for which $d(y'(a))$ and $d(\check{y}(a))$ both are $> \|dp-\overline{y}(a)\|$, there exists an $a' \in A_d$ such that $d(y'(a')) > \|dp-\overline{y}(a')\|$, $d(\check{y}(a')) > \|dp-\overline{y}(a')\|$, $\|dp-\overline{y}(a')\| \leq \frac{1}{2}\|dp-\overline{y}(a)\|$ and $d(a',a) \leq M\cdot\|dp-\overline{y}(a)\|/2e$.

Proof. As $y'(a)$ and $\check{y}(a)$ are in $B(ed,dp)$, then $u = \| dp-\bar{y}(a)\| = \| \check{y}(a)-y'(a)\|$ is $\leq 2\ ed$, thus $(2ed/u)(\check{y}(a)-y'(a))+y'(a) = \bar{h} \in B(3ed,dp)$, and there is a point $y'(a'')$, $a'' \in A_d$, by ( ), such that $\| y'(a'')-\bar{h}\| \leq ed/8$. Now let $a' \in A_d$ have the properties of (B) for $k = u/2ed$ and $\varepsilon = edk/8$.

As $k\bar{h}+(1-k)y'(a) = \check{y}(a)$, $ky'(a'')+(1-k)y'(a)$ is at a distance $\leq k\cdot ed/8$ from $y(a)$, thus:

$$(b_1) \qquad \| y'(a')-\check{y}(a)\| \leq 2\cdot edk/8 = u/8$$

Now (D) and (B_2) implies that

$$(b_2) \qquad \| \bar{y}(a')-y'(a')-(\bar{y}(a)-y'(a))\| \leq e(d)\cdot Mkd$$

and $e(d)Mkd \leq u/8$. Then $\| \bar{y}(a')-y'(a')-(\bar{y}(a)-y'(a))-(\check{y}(a)-y'(a'))\| = \| \bar{y}(a')-dp\| \leq u/4 \leq u/2$. Next, ($b_1$) implies that $dy'(a') \geq 7u/8 > u/2$, and thus $d(\check{y}(a')) > u/2$ since $\| \check{y}(a')-y'(a')\| = \| \bar{y}(a')-dp\| \leq u/4$. By ($B_2$) $d(a',a) \leq Mu/2e$, and the proof of the sublemma is finished.

Now by ($\mathcal{C}$), there is a $y'(a_o)$, $a_o \in A_d$ such that $\| dp-y'(a_o)\| < ed/4$, that is, $d(y'(a_o)) > 3ed/4$. If we let $a' = \bar{a}$, $a_o = a$ in (D) we get, by (A)

$$(a) \qquad \| \bar{y}(a_o)-y'(a_o)\| \leq \check{e}(d)\cdot Md$$

and $\check{e}(d)\cdot Md \leq ed/4$. This gives that $\| \check{y}(a_o)-dp\| = \| y'(a_o)-\bar{y}(a)\| \leq ed/4$, and

$\| dp-\bar{y}(a_o)\| < ed/2$, (as $dp-\bar{y}(a_o) = dp-y'(a_o)+y'(a_o)-\bar{y}(a_o)$). This implies that both $d(y'(a_o))$ and $d(\check{y}(a_o))$ are $> \| dp-\bar{y}(a_o)\|$.

By the sublemma, we may now find by induction a sequence $a_o, a_1, a_2, \dots$, such that for each $n \geq 1$: $\| dp-\bar{y}(a_n)\| = \frac{1}{2}\| dp-\bar{y}(a_{n-1})\|$, $d(a_n, a_{n-1}) \leq \| dp-\bar{y}(a_{n-1})\| \cdot M/2e$; and both $d(y'(a_n))$ and $d(\check{y}(a_n)) > \| dp-\bar{y}(a_n)\|$, such that the process of induction may be continued indefinitely. As $\| dp-\bar{y}(a_n)\| = (\frac{1}{2})^n \cdot \| dp-\bar{y}(a_o)\| \leq e/2^n$ we see that $\{a_n\}$ is a Cauchy sequence. Let $a_n \rightarrow a \in \bar{A}_d$. Then $\check{y}(a) = dp$ by continuity. q.e.d.

<u>The convexity property of switching</u>. Let $J' = [0,1\rangle$. Let $\mathcal{A}$ be the set of finite unions of disjoint intervals of type $[a,b\rangle$ in J'. If $h,h' \in \mathcal{L}_1(J,X)$, $k \in [0,1]$, there is, for each $\varepsilon > 0$ a set $C_k \in \mathcal{A}$ such that $\text{meas}(C_k) = k$ and

$$(7) \qquad \left\| \int_{J'} kh+(1-k)h'd\mu - \int_{J'} h\text{-}C_k+h'\cdot(1-C_k)d\mu \right\| \leq \varepsilon$$

hence

$$(8) \qquad \left\| \int_{J'} h\cdot C_k+h'(1-C_k) \right\| \leq \varepsilon + \left\| \int_{J'} kh+(1-k)h' \right\|$$

and, likewise, if $(h_1,h_1'),\dots(h_n,h_n')$ is a finite collection of pairs, we may find <u>one</u> C_k such that (a) is fulfilled for all indices $i = 1,\dots,n$). ((8) is easily seen to hold for piecewise constant functions, even for $\varepsilon = 0$.

The general case is proved by approximating h and h' by piecewise constant functions. Compare [11] Sec.II lemma 1.)

<u>Proof of the theorem</u>. Observe that $\|f'-f\|^+ \leq M\sigma(f',f)$. If $d_o > 0$ is so chosen that $B(\bar{f},d_o) \cap co\mathfrak{F} \subset N$, $y(f)$ is defined for $f \in B(\bar{f},d_o) \cap co\mathfrak{F}$. Hence, we shall prove that the system $(\tilde{\mathfrak{F}},\sigma)$, $\tilde{\mathfrak{F}}_d = B(\bar{f},d) \cap \tilde{\mathfrak{F}}$, $d \in (0,d_o]$, $\bar{f}$, $y(f)$ and $y'(f)$ fulfil the conditions of the Proposition. The above observation gives that (6) implies (D), for $\check{e}(d) = e(d) \cdot M$, and continuity of $y(f)$. To establish completeness it suffices to consider Cauchy sequenses $\{f_n\}$ of the type $\sigma(f_n,f_{n+1}) < \frac{1}{2}n+1$. Then there exist sets C_{n+1} such that $f_{b+1}(.,t)$ differ from $f_n(.,t)$ only for $t \in C_{n+1}$, and $meas(C_{n+1}) < 1/2^{n+1}$. If $B_n = \cup\{C_m/m \geq n+1\}$, we see that f_n differs from f_m, $m \geq n+1$ only on B_n, and $meas(B_n) < 1/2^n$. We may obviously find a $f \in X^{X \times J}$ such that for all $n, f = f_n$ on $\complement B_n$. Hence $f_n \to f$; by $(c\sigma)$ $f \in \tilde{\mathfrak{F}}$.

<u>(B)</u>. Let f'', $f \in \tilde{\mathfrak{F}}_d$, $k \in [0,1]$, $h = ky'(f'')+(1-k)y'(f)$. By formulas (5) and (7) there is a C_k such that for the element $f' = f''\cdot C_k+f\cdot(1-C_k)$, $\|h-y'(f')\| \leq \varepsilon$, there are sets C'' and C with $meas(C'')$ and $meas(C) < d$, such that f'', (resp. f), differ from $\bar{f}$ only on C'', (resp. C). Hence f' differ from $\bar{f}$ only on $C''\cdot C_k +$ $C\cdot(J-C_k)$ and by (8) we may choose C_k so that the measure of the former set is $< d$, that is, $f' \in \tilde{\mathfrak{F}}_d$. ($f'$ is element of $\tilde{\mathfrak{F}}$, by (SW)). Finally

$\sigma(f',f) \leq \int_0^1 (C''+C)\cdot C_k$ and by (8) we may choose C_k also so that $\sigma(f',f) \leqslant 2kd$, and (B) is proved.

(C). For any k slightly less than d, (B_1) says that $ky'(f'')+(1-k)y'(\bar{f}) = ky'(f'')$ may be approximated as closely as wanted by an element $f' = f''\cdot C_k+\bar{f}\cdot(1-C_k) \in \tilde{\mathcal{F}}$. As $\text{meas}(C_k) = k < d, f' \in \tilde{\mathcal{F}}_d$. Thus $ky'(f'') \in \overline{y'(\mathcal{F}_d)}$, hence also $dy'(f'')$ is, for any f'' in $\tilde{\mathcal{F}}$, and (C) follows. This ends the proof of the theorem.

Remark. Let L' be a line through the origin in X, let $l \in X$, and $\varphi \in X^*$ be nonzero on L'. Define $L = L'+l$. Let an admissible pair mean a system pair $(\dot{x}(.),f)$ such that $x(1) \in L$. Suppose $(\dot{\bar{x}}(.),\bar{f})$ is optimal in the problem of minimizing $\varphi(x(1))$ as function of pairs in the set of all admissible pairs. Assume now:

(body) $\qquad \text{int } K \neq \emptyset,\ K = \overline{\text{co}}\{q_f(1)/f \in \tilde{\mathcal{F}}\}$.

If we assume by contradiction that $L'^- = \{x/x \in L', \varphi(x) < 0\}$ has points in common with int K, the above theorem implies that a point on $\bar{x}(1)+L'^- \subset L$ is reachable, contradicting optimality. Thus L'^- has to be weakly separated from int K, thus also from K. If $p^* \in X^*, p^* \neq 0$, is so chosen that $p^*(K) \leq p^*(L'^-)$ we get the following

Maximum principle. If $(\dot{\bar{x}}(.),\bar{f})$ is optimal in the sence above and (body) holds, there is a $p^* \in X^*$, $p^* \neq 0$, $p^* = \alpha\varphi$ on L', $\alpha \leq 0$, and, (max'): $p^*(K) \leq 0$.

The property (max') may be rewritten, as is wellknown, in the following way

$$(\mathrm{max}) \qquad \sup_{f \in \mathcal{F}} \int_J \langle f(\bar{x}(.),.), p(.)\rangle \, d\mu = \int_J \langle \bar{f}(\bar{x}(.),.), p(.)\rangle \, d\mu$$

where $p(.)$ is the solution of

$\dot{p}(.) = -f_x^*(\bar{x}(.),.)[p(.)]$ a.e. $p(1) = p^*$.

($\dot{p}(.) \in \mathcal{L}_1(J,X^*)$, $f_x^*(\bar{x}(t),t)$ meaning the adjoint of $f_x(\bar{x}(t),t)$) (compare [12] Ch.18, p.377).

Details of the arguments in this paper, and various generalizations may be found in [13].

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